OpenStax College Algebra (Sec. 7.8) — Matrices, Determinants & Cramer's Rule

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Section 7.8 — Solving Systems with Cramer's Rule (2×2 Systems)

Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables. Consider a system of two linear equations in two variables: $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$. When $D \neq 0$, the solution using Cramer's Rule is given as:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ D \neq 0; \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ D \neq 0$$

[Page 1 · Eq #1] 5-step mathematical derivation. Step 1: x. Step 2: equals D sub x over D. Step 3: equals the determinant of the 2 by 2 matrix, Row 1: c sub 1, b sub 1; Row 2: c sub 2, b sub 2 over the determinant of the 2 by 2 matrix, Row 1: a sub 1, b sub 1; Row 2: a sub 2, b sub 2, D is not equal to 0; y. Step 4: equals D sub y over D. Step 5: equals the determinant of the 2 by 2 matrix,

Row 1: a sub 1, c sub 1; Row 2: a sub 2, c sub 2 over the determinant of the 2 by 2 matrix, Row 1: a sub 1, b sub 1; Row 2: a sub 2, b sub 2, D is not equal to 0.

Example 2 — Using Cramer's Rule to Solve a 2×2 System: Solve $12x + 3y = 15$ and $2x - 3y = 13$ using Cramer's Rule. Solution: Solve for x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

[Page 1 · Eq #2] 5-step derivation, concluding with negative 84 over negative 42, equals 2. Step 1: x. Step 2: equals D sub x over D. Step 3: equals the determinant of the 2 by 2 matrix, Row 1: 15, 3; Row 2: 13, negative 3 over the determinant of the 2 by 2 matrix, Row 1: 12, 3; Row 2: 2, negative 3. Step 4: equals negative 45 minus 39 over negative 36 minus 6. Step 5: equals negative 84 over negative 42, equals 2.

Solve for y:

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{156 - 30}{-36 - 6} = -\frac{126}{42} = -3$$

[Page 1 · Eq #3] 5-step derivation, concluding with negative 126 over 42, equals negative 3. Step 1: y. Step 2: equals D sub y over D. Step 3: equals the determinant of the 2 by 2 matrix, Row 1: 12, 15; Row 2: 2, 13 over the determinant of the 2 by 2 matrix, Row 1: 12, 3; Row 2: 2, negative 3. Step 4: equals 156 minus 30 over negative 36 minus 6. Step 5: equals negative 126 over 42, equals negative 3.

The solution is $(2, -3)$. Try It #1: Use Cramer's Rule to solve the 2×2 system $x + 2y = -11$ and $-2x + y = -13$.

Evaluating the Determinant of a 3×3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Find the determinant of the 3×3 matrix:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

[Page 1 · Eq #4] A, equals the 3 by 3 matrix, Row 1: a sub 1, b sub 1, c sub 1; Row 2: a sub 2, b sub 2, c sub 2; Row 3: a sub 3, b sub 3, c sub 3

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3×3 Determinant Expansion & Worked Example 3 (Page 2)

1. Augment A with the first two columns. 2. From upper left to lower right: multiply entries down each of the three diagonals and add. 3. From lower left to upper right: subtract the product of entries up each of the three diagonals. The algebra is as follows:

$$|A| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

[Page 2 · Eq #5] the determinant of A, equals a sub 1 b sub 2 c sub 3 plus b sub 1 c sub 2 a sub 3 plus c sub 1 a sub 2 b sub 3 minus a sub 3 b sub 2 c sub 1 minus b sub 3 c sub 2 a sub 1 minus c sub 3 a sub 2 b sub 1

Example 3 — Finding the Determinant of a 3×3 Matrix: Find the determinant of the 3×3 matrix given:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

[Page 2 · Eq #6] A, equals the 3 by 3 matrix, Row 1: 0, 2, 1; Row 2: 3, negative 1, 1; Row 3: 4, 0, 1

Solution: Augment the matrix with the first two columns and follow the diagonal formula:

$$|A| = 0(-1)(1) + 2(1)(4) + 1(3)(0) - 4(-1)(1) - 0(1)(0) - 1(3)(2) = 0 + 8 + 0 + 4 - 0 - 6 = 6$$

[Page 2 · Eq #7] 4-step derivation, concluding with 6. Step 1: the absolute value of A. Step 2: equals 0 times negative 1 times 1 plus 2 times 1 times 4 plus 1 times 3 times 0 minus 4 times negative 1 times 1 minus 0 times 1 times 0 minus 1 times 3 times 2. Step 3: equals 0 plus 8 plus 0 plus 4 minus 0 minus 6. Step 4: equals 6.

Try It #2: Find the determinant of the 3×3 matrix:

$$\det(A) = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

[Page 2 · Eq #8] the determinant of A, equals the determinant of the 3 by 3 matrix, Row 1: 1, negative 3, 7; Row 2: 1, 1, 1; Row 3: 1, negative 2, 3

Q & A: Can we use the same method to find the determinant of a larger matrix? No, this diagonal rule only works for 2×2 and 3×3 matrices. For larger matrices, use cofactor expansion or linear algebra software.

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