MATH 241 · ORIGINAL PUBLIC HANDOUT · ACCESSIBLE TAGGED PDF/UA-1 EDITION

MATH 241 · Original Public Handout — Double Integrals, Polar Coordinates & Jacobians

Accessibility Modality Note: This Tagged PDF/UA-1 file provides high-resolution vector equations paired with step-by-step spoken Alt descriptions for print and LMS archival use. For interactive symbol-by-symbol screen-reader navigation (NVDA + MathCAT, JAWS, VoiceOver), open the companion deliverable_accessible.html or deliverable_accessible.epub file included in this package.

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1. Area of an Ellipse via Linear Change of Variables

When integrating over a non-rectangular region in the plane, a smooth coordinate transformation maps a complicated boundary onto a simpler domain. Consider the interior of the ellipse $(x/a)^2 + (y/b)^2 < 1$ with semi-axes $a > 0$ and $b > 0$. Setting $x = au$ and $y = bv$ transforms the ellipse into the unit disk $u^2 + v^2 < 1$, scaling every area element by the constant factor ab:

$$\iint_{(x/a)^2+(y/b)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\,du\,dv = \pi ab$$

[Page 1 · Eq #1] 3-step derivation, concluding with pi a b. Step 1: the double integral over the quantity x over a squared, plus the quantity y over b squared is less than 1, of d x d y. Step 2: equals the double integral, over u squared plus v squared is less than 1 of a b, d u d v. Step 3: equals pi a b.

Notice the grouping in the boundary expression: $(y/b)^2$ denotes the square of the entire quotient y/b, which equals y^2/b^2 and must not be confused with y/b^2. Preserving

parenthesis scope is essential both symbolically and in spoken mathematics.

Theorem 1 (Change of Variables in Two Dimensions): Suppose $T(u, v) = (x(u, v), y(u, v))$ is a continuously differentiable bijection mapping a region S onto R. Then the local linear approximation of the coordinate differentials is governed by the Jacobian matrix:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

[Page 1 · Eq #2] the 2 by 1 column matrix, Delta u, Delta v, is approximately equal to the 2 by 2 matrix, Row 1: u sub x, u sub y; Row 2: v sub x, v sub y; times the 2 by 1 column matrix, Delta x, Delta y

For any integrable scalar field $f(x, y)$, the area element transforms by the absolute value of the Jacobian determinant:

$$\iint_R f(x, y)\, dx\, dy = \iint_S f\big(x(u, v),\, y(u, v)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

[Page 1 · Eq #3] the double integral over R, of f of x, y, d x d y, equals the double integral over S, of f of x of u, v, and y of u, v, times the absolute value of partial of x, y over partial of u, v d u d v

2. Polar Coordinates & the Polar Jacobian

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $r > 0$ and $0 \le \theta < 2\pi$, the Jacobian determinant evaluates directly from the partial derivatives:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

[Page 1 · Eq #4] 4-step derivation, concluding with r. Step 1: partial of x, y over partial of r, theta. Step 2: equals the determinant of the 2 by 2 matrix, Row 1: cosine of theta, negative r sine of

theta; Row 2: sine of theta, r cosine of theta. Step 3: equals r cosine squared theta plus r sine squared theta. Step 4: equals r.

Source Lecture Page 2

3. Worked Example: Definite Trigonometric & Gaussian Integrals

Evaluating definite integrals with explicit upper and lower bracket limits requires careful tracking of antiderivative coefficients. Consider the oscillatory integral over the first eighth-turn:

$$\int_0^{\pi/4} \cos(2\theta)\, d\theta = \left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = \frac{1}{2}$$

[Page 2 · Eq #5] the integral from 0 to pi over 4, of cosine of 2 theta, d theta, equals open bracket 1 over 2 sine of 2 theta close bracket, evaluated from 0 to pi over 4, equals 1 over 2

A celebrated application of the polar Jacobian (4) is the exact evaluation of the one-dimensional Gaussian integral. Squaring the integral turns the product of two independent lines into a radially symmetric double integral over $\mathbb{R}^2$:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = \pi$$

[Page 2 · Eq #6] 4-step derivation, concluding with pi. Step 1: I squared. Step 2: equals open paren the integral from negative infinity to infinity, of e to the negative x squared power, d x close paren squared. Step 3: equals the integral from 0 to 2 pi, of the integral from 0 to infinity, of e to the negative r squared power, times r d r d theta. Step 4: equals pi.

Taking the positive square root yields the fundamental probability normalization identity:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

[Page 2 · Eq #7] the integral from negative infinity to infinity, of e to the negative x squared power, d x, equals the square root of pi

4. Summary of Common Spoken-Math Pitfalls

When mathematical notation is read aloud by standard screen readers from an untagged PDF, flat text extraction frequently drops fraction bars, collapses 2D matrices into single number streams, and merges exponent scope. In particular, distinguish carefully between:

$$\left(\frac{y}{b}\right)^2 = \frac{y^2}{b^2} \neq \frac{y}{b^2}$$

[Page 2 · Eq #8] the quantity y over b squared, equals y squared, over b squared is not equal to y, over b squared

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