Change of Variables, Polar Integrals & Jacobian Determinants

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1. Area of an Ellipse via Linear Change of Variables

When integrating over a non-rectangular region in the plane, a smooth coordinate transformation maps a complicated boundary onto a simpler domain. Consider the interior of the ellipse $(x/a)^2 + (y/b)^2 < 1$ with semi-axes $a > 0$ and $b > 0$. Setting $x = au$ and $y = bv$ transforms the ellipse into the unit disk $u^2 + v^2 < 1$, scaling every area element by the constant factor ab:

$$\iint_{(x/a)^2+(y/b)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\,du\,dv = \pi ab \tag{1}$$

Notice the grouping in the boundary expression: $(y/b)^2$ denotes the square of the entire quotient y/b, which equals y^2/b^2 and must not be confused with y/b^2. Preserving parenthesis scope is essential both symbolically and in spoken mathematics.

> **Theorem 1 (Change of Variables in Two Dimensions).**
> Suppose $T(u, v) = (x(u,v),\ y(u,v))$ is a continuously differentiable bijection mapping a region S onto R. Then the local linear approximation of the coordinate differentials is governed by the Jacobian matrix:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{2}$$

For any integrable scalar field $f(x, y)$, the area element transforms by the absolute value of the Jacobian determinant:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f\big(x(u,v),\ y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \tag{3}$$

2. Polar Coordinates & the Polar Jacobian

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $r > 0$ and $0 \le \theta < 2\pi$, the Jacobian determinant evaluates directly from the partial derivatives:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \tag{4}$$

3. Worked Example: Definite Trigonometric & Gaussian Integrals

Evaluating definite integrals with explicit upper and lower bracket limits requires careful tracking of antiderivative coefficients. Consider the oscillatory integral over the first eighth-turn:

$$\int_{0}^{\pi/4} \cos(2\theta)\, d\theta = \left[\frac{1}{2}\sin(2\theta) \right]_{0}^{\pi/4} = \frac{1}{2} \tag{5}$$

A celebrated application of the polar Jacobian (4) is the exact evaluation of the one-dimensional Gaussian integral. Squaring the integral turns the product of two independent lines into a radially symmetric double integral over $\mathbb{R}^2$:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2}\, r\, dr\, d\theta = \pi \tag{6}$$

Taking the positive square root yields the fundamental probability normalization identity:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{7}$$

4. Summary of Common Spoken-Math Pitfalls

When mathematical notation is read aloud by standard screen readers from an untagged PDF, flat text extraction frequently drops fraction bars, collapses 2D matrices into single number streams, and merges exponent scope. In particular, distinguish carefully between:

$$\left(\frac{y}{b}\right)^2 = \frac{y^2}{b^2} \quad \neq \quad \frac{y}{b^2} \tag{8}$$

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