11. $\int_0^\infty e^{-x} x^n dx = \lfloor n$ and hence

$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) - \&c\right\} dx = \lfloor n-1\ \phi(-n)$$

Sol. $\int_0^\infty e^{-x} x^n dx = e^{-x}\{x^n + n x^{n-1} + n(n-1)x^{n-2} + \&c\}$

when $x = 0 = \lfloor n$ by IV 10 Cor.

$$f(0)\int_0^\infty e^{-x} x^{n-1} dx = \lfloor n-1 f(0)$$

$$\frac{f'(0)}{\lfloor 1}\int_0^\infty e^{-\pi x} x^{n-1} dx = \frac{\lfloor n-1}{\pi^n}\cdot\frac{f'(0)}{\lfloor 1}$$

$$\frac{f''(0)}{\lfloor 2}\int_0^\infty e^{-\pi^2 x} x^{n-1} dx = \frac{\lfloor n-1}{\pi^{2n}}\cdot\frac{f''(0)}{\lfloor 2}$$

and so on.

Adding up all the results we have

$$\int_0^\infty x^{n-1}\left\{f(1) - \frac{x}{\lfloor 1}f(\pi) + \frac{x^2}{\lfloor 2}f(\pi^2) - \&c\right\} dx = \lfloor n-1\ f\left(\frac{1}{\pi^n}\right).$$

Let $f(\pi^n) = \phi(n)$ then $f\left(\frac{1}{\pi^n}\right) = \phi(-n)$.

Cor 1. $\int_0^\infty x^{n-1}\left\{\phi(0) - x\phi(1) + x^2\phi(2) - \&c\right\} dx = \frac{\pi\,\phi(-n)}{\sin \pi n}$

Cor 2. $\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x^2}{\lfloor 2}\phi(2) + \frac{x^4}{\lfloor 4}\phi(4) - \&c\right\} dx = \lfloor n-1\ \phi(-n) \times \cos\frac{\pi n}{2}$

Cor 3. $\int_0^\infty \left\{\phi(0) - \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) - \&c\right\} \cos nx\, dx$

$= \phi(-1) - n^2\phi(-3) + n^4\phi(-5) - \&c.$

Cor 4. $\int_0^\infty \left\{\phi(0) - x^2\phi(2) + x^4\phi(4) - \&c\right\} \cos nx\, dx$

$= \frac{\pi}{2}\left\{\phi(-1) - \frac{n}{\lfloor 1}\phi(-2) + \frac{n^2}{\lfloor 2}\phi(-3) - \frac{n^3}{\lfloor 3}\phi(-4) + \&c\right\}$